Proper factorization systems in 2-categories

M. Dupont E.M. Vitale

4th July 2002

Abstract. Starting from known examples of factorization systems in 2-categories, we discuss possible definitions of proper factorization system in a 2-category. We focus our attention on the construction of the free proper factorization system on a given 2-category.

MSC 2000 : 18A20, 18A32, 18D05.

Introduction

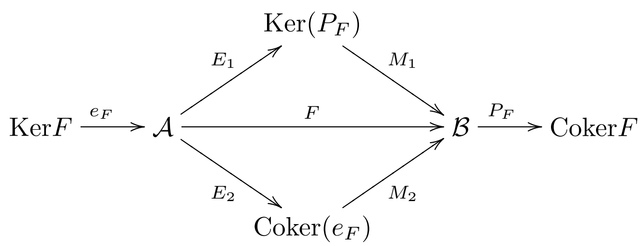
The notion of factorization system in a category is well established and has a lot of applications to basic category theory [7] as well as to some more specific topic, like categorical topology [11] or categorical Galois theory [8]. When a relevant construction emerges in mathematics the question of existence of free such structure is always important. In [20], M. Korostenski and W. Tholen study the free category with factorization system on a given category $\mathcal{C}$. They prove that it is given by the embedding $\mathcal{C} \to \mathcal{C}^2$ of $\mathcal{C}$ into its category of morphisms. In general, given a factorization system $(\mathcal{E}, \mathcal{M})$ in a category and the corresponding factorization $f = (m \in \mathcal{M}) \circ (e \in \mathcal{E})$ of an arrow f, it is a common intuition to think to e as the "surjective" part of f and to m as the "injective" part of f. This is the case for the standard factorization system in Set, as well as for many other natural examples, but it is by no way a consequence of the definition of factorization system. A factorization system such that the class $\mathcal{E}$ is contained in the class of epimorphisms and the class $\mathcal{M}$ in that of monomorphisms is called proper. The free category $\mathrm{Fr}\mathcal{C}$ with proper factorization system on a given category $\mathcal{C}$ has been studied by M. Grandis in [15], where it is proved that $\mathrm{Fr}\mathcal{C}$ is a quotient of $\mathcal{C}^2$, so that we can picture the situation with the diagram

$$\mathcal{C} \longrightarrow \mathcal{C}^2 \longrightarrow \mathrm{Fr}\mathcal{C}\,.$$

The category $\mathrm{Fr}\mathcal{C}$ is of special interest for its applications to the stable homotopy category (in this case it is also called the Freyd completion of $\mathcal{C}$, which explains the notation), to homology theories and to triangulated categories (see [5, 10, 13, 14, 23, 25]).

For the needs of 2-dimensional homological algebra, S. Kasangian and the second author introduced in [17] the notion of factorization system in a 2-category with invertible 2-arrows, showing the existence of two such factorization systems in the 2-category SCG of symmetric categorical groups. Subsequently, the definition has been extended by S. Milius to arbitrary 2-categories in [22], where the basic theory is developed. In particular, Milius exhibits the free 2-category with factorization system $\mathbb{C} \to \mathbb{C}^2$ on a given 2-category $\mathbb{C}$, which is the 2-dimensional analogue of the Korostenski-Tholen construction. The aim of this note is to complete the picture, giving the 2-dimensional analogue of Grandis construction, that is the free 2-category with proper factorization system.

For this, let us look more carefully at the two factorization systems for symmetric categorical groups discussed in [17]. In the first one, an arrow F factors through the kernel of its cokernel; in the second one it factors through the cokernel of its kernel



and one has that E_1 is full and essentially surjective, M_1 is faithful, E_2 is essentially surjective and M_2 is full and faithful.
Now, for a morphism F in SCG (that is, F is a monoidal functor compatible with the symmetry), one has the following situation:

- F is faithful (respectively, full and faithful) iff for any $\mathcal{G} \in$ SCG, the hom-functor SCG$(\mathcal{G}, F)\colon$ SCG$(\mathcal{G}, \mathcal{A}) \to$ SCG$(\mathcal{G}, \mathcal{B})$ is faithful (respectively, full and faithful);

- F is essentially surjective (respectively, full and essentially surjective) iff for any $\mathcal{G} \in$ SCG, the hom-functor SCG$(F, \mathcal{G})\colon$ SCG$(\mathcal{B}, \mathcal{G}) \to$ SCG$(\mathcal{A}, \mathcal{G})$ is faithful (respectively, full and faithful).

This situation suggests to analyze the following variants of the notion of proper factorization system in a 2-category $\mathbb{C}$: a factorization system $(\mathcal{E}, \mathcal{M})$ is

- (1,1)-proper if for any $f \in \mathcal{M}$ the hom-functors $\mathbb{C}(X, f)$ are faithful and for any $f \in \mathcal{E}$ the hom-functors $\mathbb{C}(f, X)$ are faithful (with X varying in $\mathbb{C}$);

- (2,1)-proper if it is (1,1)-proper and moreover for any $f \in \mathcal{E}$ the hom-functors $\mathbb{C}(f, X)$ are full;

- (1,2)-proper if it is (1,1)-proper and moreover for any $f \in \mathcal{M}$ the hom-functors $\mathbb{C}(X, f)$ are full;

- (2,2)-proper if it is (2,1)-proper and (1,2)-proper, i.e. if for any $f \in \mathcal{M}$ the hom-functors $\mathbb{C}(X, f)$ are fully faithful and for any $f \in \mathcal{E}$ the hom-functors $\mathbb{C}(f, X)$ are fully faithful.

For these four kinds of proper factorization systems, we give the construction of the free 2-category with proper factorization system on a given 2-category $\mathbb{C}$. The situation can be summarized in the following diagram (where $\mathrm{Fr}^{i,j}\mathbb{C}$ is the free 2-category with (i,j)-proper factorization system)

$$\begin{array}{ccccccccc} & & & & & & \mathrm{Fr}^{1,2}\mathbb{C} & & \\ & & & & & \nearrow & & \searrow & \\ \mathbb{C} & \longrightarrow & \mathbb{C}^2 & \longrightarrow & \mathrm{Fr}^{1,1}\mathbb{C} & & & & \mathrm{Fr}^{2,2}\mathbb{C} \\ & & & & & \searrow & & \nearrow & \\ & & & & & & \mathrm{Fr}^{2,1}\mathbb{C} & & \end{array}$$

(conditions on $\mathbb{C}$ are needed to define $\mathrm{Fr}^{2,2}\mathbb{C}$, see Section 6).

The embedding $\mathcal{C} \longrightarrow \mathrm{Fr}\mathcal{C}$ is a step in the construction of the free regular, exact or abelian category on $\mathcal{C}$ (see [21, 25]). From this point of view, the present paper is part of a program devoted to study similar notions for 2-categories, and it is intended to clarify the delicate notions of monomorphism and epimorphism in a 2-categorical setting (see also [1, 3, 6, 9, 18, 26]).

The paper is organized as follows. In Section 1 we give the definition of factorization system in a 2-category as it appears in [12, 22]. It is slightly different from that given in [17], but they are equivalent if the 2-cells are invertible. In Section 2 we recall, from [22], the construction of the free 2-category with factorization system. In Sections 3 we fix the terminology for arrows in a 2-category. In Sections 4, 5 and 6 we describe the various $\mathrm{Fr}^{i,j}\mathbb{C}$ and we prove their universal property. Section 7 is devoted to examples and to an open problem. Finally, in Section 8, we give a glance at the relation between factorization systems in 2-categories and in categories. If $\mathbb{C}$ is a locally discrete 2-category (that is, a category), then our definition coincide with the usual definition of factorization system. But a factorization system in a 2-category $\mathbb{C}$ does not induce a factorization system (in the usual sense) neither in the underlying category of $\mathbb{C}$ nor in the homotopy category $H(\mathbb{C})$ of $\mathbb{C}$. The best we can say is that it induces in $H(\mathbb{C})$ a weak factorization system (a structure of interest especially for Quillen approach to homotopy theory, see [2, 4, 16, 24]), and even this fact is not completely obvious to prove.

1 Factorization systems in 2-categories

To define the notion of factorization system in a 2-category, we need the orthogonality condition. A first 2-categorical version of this condition was introduced in [17] for a 2-category with invertible 2-cells. Since we work in an arbitrary 2-category, we need a stronger version, as in [12, 22].

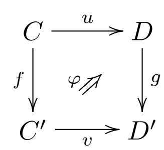
Definition 1.1. Let $\mathbb{C}$ be a 2-category and consider two arrows $f\colon C \longrightarrow C'$ and $g\colon D \longrightarrow D'$ in $\mathbb{C}$. We say that *f is orthogonal to g*, denoted by $f \downarrow g$, if the following diagram is a bi-pullback in Cat

$$\begin{array}{ccc} \mathbb{C}(C',D) & \xrightarrow{-\circ f} & \mathbb{C}(C,D) \\ {\scriptstyle g\circ -}\downarrow & & \downarrow{\scriptstyle g\circ -} \\ \mathbb{C}(C',D') & \xrightarrow[-\circ f]{} & \mathbb{C}(C,D') \end{array}$$

If $\mathcal{H}$ is a class of arrows of $\mathbb{C}$, we write $\mathcal{H}^{\uparrow} = \{e \mid e \downarrow h \text{ for all } h \in \mathcal{H}\}$ and $\mathcal{H}^{\downarrow} = \{m \mid h \downarrow m \text{ for all } h \in \mathcal{H}\}$.

To make the previous definition more explicit, we need some point of terminology.

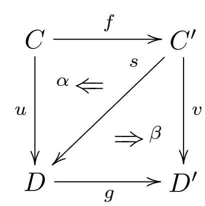
Definition 1.2. The *2-category of arrows* of $\mathbb{C}$, denoted by $\mathbb{C}^2$, is the 2-category whose objects are arrows of $\mathbb{C}$, whose 1-cells are triples (u, φ, v), as in the following diagram, where φ is invertible,



and whose 2-cells $(u, \varphi, v) \Rightarrow (w, \psi, x) : f \longrightarrow g$ are pairs (α, β) of 2-cells of $\mathbb{C}$, with $\alpha : u \Rightarrow w$ and $\beta : v \Rightarrow x$ such that

$$(g * \alpha) \circ \varphi = \psi \circ (\beta * f).$$

Definition 1.3. Let (u, φ, v) be an arrow from f to g in $\mathbb{C}^2$. A *fill-in* for (u, φ, v) is a triple (α, s, β), as in the following diagram, with $\alpha : sf \Rightarrow u$ and $\beta : gs \Rightarrow v$ invertible and such that $g * \alpha = \varphi(\beta * f)$.



The fill-in (α, s, β) is *universal* if for any other fill-in (γ, t, δ) for (u, φ, v), there is a unique invertible $\omega : t \Rightarrow s$ such that $\gamma = \alpha(\omega * f)$ and $\delta = \beta(g * \omega)$.

Proposition 1.4. *Let $f : C \longrightarrow C'$ and $g : D \longrightarrow D'$ be two arrows in a 2-category $\mathbb{C}$. Then $f \downarrow g$ if and only if the following conditions hold:*

1. *each morphism $(u, \varphi, v) : f \longrightarrow g$ has a universal fill-in;*

2. *for each $(u, \varphi, v), (u', \varphi', v') : f \longrightarrow g$, for each $(\mu, \nu) : (u, \varphi, v) \Rightarrow (u', \varphi', v')$ in $\mathbb{C}^2$, for each universal fill-in (α, s, β) and (α', s', β') respectively for (u, φ, v) and for (u', φ', v'), there is a unique $\sigma : s \Rightarrow s'$ such that*

$$\mu \circ \alpha = \alpha' \circ (\sigma * f) \qquad \textit{and} \qquad \nu \circ \beta = \beta' \circ (g * \sigma). \tag{1}$$

The former version of the orthogonality condition, in [17], consists only of condition 1 of the previous proposition. When all 2-cells are invertible, condition 2 follows from condition 1.

The following lemma is sometimes useful to check the orthogonality condition.

Lemma 1.5. 1. If there exists a universal fill-in for $(u, \varphi, v) : f \longrightarrow g$, then every fill-in for (u, φ, v) is universal.

2. $f \downarrow g$ if and only if

   (a) each morphism $(u, \varphi, v) : f \longrightarrow g$ has a fill-in;

   (b) for each $(u, \varphi, v), (u', \varphi', v') : f \longrightarrow g$, for each $(\mu, \nu) : (u, \varphi, v) \Rightarrow (u', \varphi', v')$, for each fill-in (α, s, β) and (α', s', β') respectively for (u, φ, v) and for (u', φ', v'), there is a unique $\sigma : s \Rightarrow s'$ such that equations (1) hold.

Here is the definition of a factorization system in $\mathbb{C}$.

Definition 1.6. A *factorization system* in a 2-category $\mathbb{C}$ is a pair $(\mathcal{E}, \mathcal{M})$ of classes of arrows in $\mathbb{C}$ such that:

1. if $m \in \mathcal{M}$ and i is an equivalence then $mi \in \mathcal{M}$, and if $e \in \mathcal{E}$ and i is an equivalence then $ie \in \mathcal{E}$;

2. $\mathcal{E}$ and $\mathcal{M}$ are stable under invertible 2-cells (i.e. if $e \in \mathcal{E}$ and $\alpha : f \Rightarrow e$ is invertible, then $f \in \mathcal{E}$, and the same property holds for $\mathcal{M}$);

3. for each arrow f in $\mathbb{C}$, there exists $e \in \mathcal{E}$, $m \in \mathcal{M}$ and an invertible 2-cell $\varphi : me \Rightarrow f$ (such a factorization φ of f is called an $(\mathcal{E}, \mathcal{M})$-*factorization* of f);

4. for each $e \in \mathcal{E}$ and for each $m \in \mathcal{M}$, $e \downarrow m$.

The proof of the basic properties of factorization systems in 2-categories can be found in [17] and [22].

Proposition 1.7. *Let $(\mathcal{E}, \mathcal{M})$ be a factorization system in $\mathbb{C}$. The following properties hold.*

1. *$\mathcal{E} \cap \mathcal{M} = \{equivalences\}$.*

2. *$\mathcal{E}$ and $\mathcal{M}$ are closed under composition.*

3. *$\mathcal{E} = \mathcal{M}^{\uparrow}$ and $\mathcal{M} = \mathcal{E}^{\downarrow}$.*

4. *The $(\mathcal{E}, \mathcal{M})$-factorization of an arrow of $\mathbb{C}$ is essentially unique (i.e. if $\varphi : me \Rightarrow f$ and $\varphi' : m'e' \Rightarrow f$ are two such factorizations, there exist an equivalence i and invertible 2-cells $\alpha : ie \Rightarrow e'$ and $\beta : m'i \Rightarrow m$ such that $\varphi \circ (\beta * e) = \varphi' \circ (m' * \alpha)$);*

5. *(Cancellation property) If $m', m \in \mathcal{M}$ and if $\theta : m'g \Rightarrow m$ is an invertible 2-cell, then $g \in \mathcal{M}$; dually, if $e', e \in \mathcal{E}$ and if $\theta : fe' \Rightarrow e$ is an invertible 2-cell, then $f \in \mathcal{E}$.*

6. *$\mathcal{M}$ is stable under bi-limits and $\mathcal{E}$ is stable under bi-colimits.*

Remark: In Definition 1.6, conditions 1, 2 and 4 can be equivalently replaced by point 3 of Proposition 1.7.

2 Free 2-categories with factorization system

In this section, we describe the free 2-category with factorization system on a given 2-category $\mathbb{C}$

$$E_{\mathbb{C}} \colon \mathbb{C} \longrightarrow \mathbb{C}^2 \,.$$

In fact, $\mathbb{C}^2$ is provided with the following factorization system $(\mathcal{E}_{\mathbb{C}}, \mathcal{M}_{\mathbb{C}})$

$$\mathcal{E}_{\mathbb{C}} = \{(u, \varphi, v) \,|\, u \text{ is an equivalence}\}$$

$$\mathcal{M}_{\mathbb{C}} = \{(u, \varphi, v) \,|\, v \text{ is an equivalence}\}.$$

An arrow $(u, \varphi, v) : f \longrightarrow g$ in $\mathbb{C}^2$ factors as in the following diagram.

$$\begin{array}{ccccc} C & \xrightarrow{1_C} & C & \xrightarrow{u} & D \\ {\scriptstyle f}\downarrow & \nearrow\!\!\!\nearrow \varphi & \downarrow{\scriptstyle gu} & & \downarrow{\scriptstyle g} \\ C' & \xrightarrow[v]{} & D' & \xrightarrow[1_{D'}]{} & D' \end{array} \tag{2}$$

We write $e_{(u,\varphi,v)} = (1_C, \varphi, v)$ and $m_{(u,\varphi,v)} = (u, 1_{gu}, 1_{D'})$. The 2-functor $E_{\mathbb{C}}$: $\mathbb{C} \longrightarrow \mathbb{C}^2$ maps an object $C \in \mathbb{C}$ to 1_C, an arrow $f \in \mathbb{C}(C, C')$ to $(f, 1_f, f)$, and a 2-cell $\alpha : f \Rightarrow g : C \longrightarrow C'$ to (α, α).

If $\mathbb{C}$ and $\mathbb{D}$ are 2-categories, we write $\mathrm{PS}(\mathbb{C}, \mathbb{D})$ for the 2-category of pseudo-functors from $\mathbb{C}$ to $\mathbb{D}$, pseudo-natural transformations and modifications. If $\mathbb{C}$ and $\mathbb{D}$ are 2-categories with factorization system, $\mathrm{PS}_{fs}(\mathbb{C}, \mathbb{D})$ is the 2-category of pseudo-functors preserving the factorization system (i.e. $F(\mathcal{E}) \subseteq \mathcal{E}$ and $F(\mathcal{M}) \subseteq \mathcal{M}$), pseudo-natural transformations and modifications. Here is the universal property of $E_{\mathbb{C}} \colon \mathbb{C} \longrightarrow \mathbb{C}^2$.

Proposition 2.1. *For each 2-category $\mathbb{C}$ and for each 2-category $(\mathbb{D}, (\mathcal{E}, \mathcal{M}))$ with factorization system, the 2-functor*

$$- \circ E_{\mathbb{C}} : \mathrm{PS}_{fs}(\mathbb{C}^2, \mathbb{D}) \longrightarrow \mathrm{PS}(\mathbb{C}, \mathbb{D})$$

is a biequivalence.

Proof. A proof can be found in [22]. For reader's convenience, we recall how to construct, from an arbitrary pseudo-functor $G \colon \mathbb{C} \longrightarrow \mathbb{D}$, a pseudo-functor $F : \mathbb{C}^2 \longrightarrow \mathbb{D}$ preserving the factorization system and such that $FE_{\mathbb{C}} \cong G$. Observe that, given an object $f \colon C \longrightarrow C'$ in $\mathbb{C}^2$, we get a commutative diagram

$$\begin{array}{ccccc} C & \xrightarrow{1_C} & C & \xrightarrow{f} & C' \\ {\scriptstyle 1_C}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 1_{C'}} \\ C & \xrightarrow[f]{} & C' & \xrightarrow[1_{C'}]{} & C' \end{array}$$

where the square on the left is an arrow in $\mathcal{E}_{\mathbb{C}}$ and the square on the right is an arrow in $\mathcal{M}_{\mathbb{C}}$. This means that f is the image of $E_{\mathbb{C}}(f)$ in $\mathbb{C}^2$. Now, if we want F to preserve the factorization system and if we want an equivalence $FE_{\mathbb{C}} \cong G$, we have to define $F(f)$ as the image of $G(f)$ in $\mathbb{D}$. The definition of F on 1-cells and on 2-cells follows now from the orthogonality condition. $\square$

3 Arrows in a 2-category

We introduce now a terminology to name various kinds of arrows in a 2-category. Our terminology will be justified by the examples $\mathbb{C} = \mathrm{Cat}$ and $\mathbb{C} = \mathrm{SCG}$ discussed in Section 7.

Definition 3.1. Let $\mathbb{C}$ be a 2-category and $f : C \longrightarrow C'$, an arrow in $\mathbb{C}$.

1. We say that f is *faithful* if for each $X \in \mathbb{C}$, the functor $f \circ - : \mathbb{C}(X, C) \longrightarrow \mathbb{C}(X, C')$ is faithful.

2. We say that f is *fully faithful* if for each $X \in \mathbb{C}$, the functor $f \circ - : \mathbb{C}(X, C) \longrightarrow \mathbb{C}(X, C')$ is fully faithful.

3. We say that f is *cofaithful* if for each $Y \in \mathbb{C}$, the functor $- \circ f : \mathbb{C}(C', Y) \longrightarrow \mathbb{C}(C, Y)$ is faithful.

4. We say that f is *fully cofaithful* if for each $Y \in \mathbb{C}$, the functor $- \circ f : \mathbb{C}(C', Y) \longrightarrow \mathbb{C}(C, Y)$ is fully faithful.

This terminology for arrows generates a terminology for factorization systems, which generalizes the term "proper factorization system" used for usual categories.

Definition 3.2. Let $(\mathcal{E}, \mathcal{M})$ be a factorization system on a 2-category $\mathbb{C}$.

1. We say that $(\mathcal{E}, \mathcal{M})$ is *(1,1)-proper* if each $e \in \mathcal{E}$ is cofaithful and each $m \in \mathcal{M}$ is faithful.

2. We say that $(\mathcal{E}, \mathcal{M})$ is *(2,1)-proper* if each $e \in \mathcal{E}$ is fully cofaithful and each $m \in \mathcal{M}$ is faithful.

3. We say that $(\mathcal{E}, \mathcal{M})$ is *(1,2)-proper* if each $e \in \mathcal{E}$ is cofaithful and each $m \in \mathcal{M}$ is fully faithful.

4. We say that $(\mathcal{E}, \mathcal{M})$ is *(2,2)-proper* if each $e \in \mathcal{E}$ is fully cofaithful and each $m \in \mathcal{M}$ is fully faithful.

Remark: If $\mathbb{C}$ is locally discrete, then any factorization system $(\mathcal{E}, \mathcal{M})$ on $\mathbb{C}$ is (1,1)-proper. It is (2,1)-proper exactly when $\mathcal{E}$ is contained in the class of epimorphisms, and (1,2)-proper when $\mathcal{M}$ is contained in the class of monomorphisms. Finally, $(\mathcal{E}, \mathcal{M})$ is (2,2)-proper exactly when it is proper in the usual sense.

In the sequel, we will construct the free 2-category with a (i, j)-proper (for $i = 1, 2, j = 1, 2$) factorization system on a given 2-category.

4 (1,1)-proper factorization systems

In this section, we describe the free 2-category with (1,1)-proper factorization system on a given 2-category $\mathbb{C}$

$$E^{1,1}_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathrm{Fr}^{1,1}\mathbb{C}\,.$$

Definition 4.1. Let $\mathbb{C}$ be a 2-category. The 2-category $\mathrm{Fr}^{1,1}\mathbb{C}$ has the same objects and arrows as $\mathbb{C}^2$, but a 2-cell between two arrows (u, φ, v) and $(w, \psi, x) : f \longrightarrow g$ is an equivalence class of 2-cells of $\mathbb{C}^2$ between the same arrows, for the equivalence relation

$$\begin{array}{lll} (\alpha, \beta) \sim (\alpha', \beta') & \text{iff} & g * \alpha = g * \alpha' \\ & \text{iff} & \beta * f = \beta' * f. \end{array}$$

We write $[\alpha, \beta]$ for the equivalence class of (α, β). The composition of 2-cells is the same as in $\mathbb{C}^2$, modulo $\sim$: $[\alpha', \beta'] \circ [\alpha, \beta] = [\alpha' \circ \alpha, \beta' \circ \beta]$ and $[\gamma, \delta] * [\alpha, \beta] = [\gamma * \alpha, \delta * \beta]$.

The 2-category $\mathrm{Fr}^{1,1}\mathbb{C}$ is equipped with a factorization system $(\mathcal{E}^{1,1}_{\mathbb{C}}, \mathcal{M}^{1,1}_{\mathbb{C}})$ which factorizes an arrow (u, φ, v) as in $\mathbb{C}^2$, diagram (2). Following the notations of (2),

$$\mathcal{E}^{1,1}_{\mathbb{C}} = \{(u, \varphi, v) \,|\, m_{(u,\varphi,v)} \text{ is an equivalence in } \mathrm{Fr}^{1,1}\mathbb{C}\}$$

$$\mathcal{M}^{1,1}_{\mathbb{C}} = \{(u, \varphi, v) \,|\, e_{(u,\varphi,v)} \text{ is an equivalence in } \mathrm{Fr}^{1,1}\mathbb{C}\}.$$

Proposition 4.2. *The factorization system* $(\mathcal{E}^{1,1}_{\mathbb{C}}, \mathcal{M}^{1,1}_{\mathbb{C}})$ *in* $\mathrm{Fr}^{1,1}\mathbb{C}$ *is (1,1)-proper.*

Proof. We have to prove that, if $(u, \varphi, v) : f \longrightarrow g$ is an arrow in $\mathrm{Fr}^{1,1}\mathbb{C}$, then $e_{(u,\varphi,v)}$ is cofaithful and $m_{(u,\varphi,v)}$ is faithful.
Let h be an object of $\mathrm{Fr}^{1,1}\mathbb{C}$. Let $[\alpha, \beta], [\alpha', \beta'] : (w, \psi, x) \Rightarrow (w', \psi', x') : gu \longrightarrow h$ be 2-cells of $\mathrm{Fr}^{1,1}\mathbb{C}$ such that

$$[\alpha, \beta] * e_{(u,\varphi,v)} = [\alpha', \beta'] * e_{(u,\varphi,v)}.$$

Since $e_{(u,\varphi,v)} = (1_C, \varphi, v) : f \longrightarrow gu$ (cf. diagram 2), this equation becomes

$$[\alpha, \beta * v] = [\alpha', \beta' * v],$$

which, by definition of $\mathrm{Fr}^{1,1}\mathbb{C}$, is equivalent to

$$h * \alpha = h * \alpha'. \tag{3}$$

This implies that $[\alpha, \beta] = [\alpha', \beta']$, since this last equation is also equivalent, by definition of $\mathrm{Fr}^{1,1}\mathbb{C}$, to equation (3). So $e_{(u,\varphi,v)}$ is cofaithful. The proof that $m_{(u,\varphi,v)}$ is faithful is similar. ☐

Consider the quotient 2-functor $P^{1,1}_{\mathbb{C}} : \mathbb{C}^2 \longrightarrow \mathrm{Fr}^{1,1}\mathbb{C}$, which is the identity on objects and arrows and maps a 2-cell (α, β) to its equivalence class $[\alpha, \beta]$. We can define the 2-functor $E^{1,1}_{\mathbb{C}} = P^{1,1}_{\mathbb{C}} \circ E_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}^2 \longrightarrow \mathrm{Fr}^{1,1}\mathbb{C}$. Its universal property is stated in the following proposition.

Proposition 4.3. *For any 2-category* $\mathbb{C}$ *and for any 2-category* $(\mathbb{D}, (\mathcal{E}, \mathcal{M}))$ *with (1,1)-proper factorization system, the 2-functor*

$$- \circ E^{1,1}_{\mathbb{C}} : \mathrm{PS}_{\mathrm{f}s}(\mathrm{Fr}^{1,1}\mathbb{C}, \mathbb{D}) \longrightarrow \mathrm{PS}(\mathbb{C}, \mathbb{D})$$

is a biequivalence.

Proof. Since $E^{1,1}_{\mathbb{C}} = P^{1,1}_{\mathbb{C}} \circ E_{\mathbb{C}}$ and since Proposition 2.1 tells us that $- \circ E_{\mathbb{C}}$ is a biequivalence, it remains to prove that

$$- \circ P^{1,1}_{\mathbb{C}} : \mathrm{PS}_{\mathrm{f}s}(\mathrm{Fr}^{1,1}\mathbb{C}, \mathbb{D}) \longrightarrow \mathrm{PS}_{\mathrm{f}s}(\mathbb{C}^2, \mathbb{D})$$

is a biequivalence (it is well-defined because $P^{1,1}_{\mathbb{C}}$ preserves the factorization system).

It is straightforward to prove that $- \circ P^{1,1}_{\mathbb{C}}$ is locally an equivalence. As far as its surjectivity up to equivalence is concerned, let $G : \mathbb{C}^2 \longrightarrow \mathbb{D}$ be a pseudo-functor preserving the factorization system. We have to find a pseudo-functor $F : \mathrm{Fr}^{1,1}\mathbb{C} \longrightarrow \mathbb{D}$ preserving the factorization system, such that $FP^{1,1}_{\mathbb{C}}$ is equivalent to G.
On objects and arrows, we take $F = G$. If $[\alpha, \beta] : (u, \varphi, v) \Rightarrow (w, \psi, x)$ is a 2-cell in $\mathrm{Fr}^{1,1}\mathbb{C}$, we take $F([\alpha, \beta]) = G(\alpha, \beta)$. Then $FP^{1,1}_{\mathbb{C}} = G$ and it remains to prove that F is well defined, i.e. if $[\alpha, \beta] = [\gamma, \delta] : (u, \varphi, v) \Rightarrow (w, \psi, x) : f \longrightarrow g$, then $G(\alpha, \beta) = G(\gamma, \delta)$.
By definition of $\mathrm{Fr}^{1,1}\mathbb{C}$, $g * \alpha = g * \gamma$ and $\beta * f = \delta * f$. So

$$G(g * \alpha, \beta * f) = G(g * \gamma, \delta * f). \quad (4)$$

But, up to invertible 2-cells, equation (4) becomes

$$G(g, 1_g, 1_{D'}) * G(\alpha, \beta) * G(1_C, 1_f, f) = G(g, 1_g, 1_{D'}) * G(\gamma, \delta) * G(1_C, 1_f, f). \quad (5)$$

Since $(g, 1_g, 1_{D'}) \in \mathcal{M}_{\mathbb{C}}$ and G preserves the factorization system, $G(g, 1_g, 1_{D'}) \in \mathcal{M}$. Since $(\mathcal{E}, \mathcal{M})$ is (1,1)-proper, $G(g, 1_g, 1_{D'})$ is faithful. Thus equation (5) is equivalent to

$$G(\alpha, \beta) * G(1_C, 1_f, f) = G(\gamma, \delta) * G(1_C, 1_f, f).$$

Similarly, $G(1_C, 1_f, f)$ is cofaithful, and we can conclude that $G(\alpha, \beta) = G(\gamma, \delta)$. □

5 (2,1)-proper and (1,2)-proper factorization systems

In this section, we describe the free 2-category with (2,1)-proper factorization system on a given 2-category $\mathbb{C}$

$$E^{2,1}_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathrm{Fr}^{2,1}\mathbb{C}\,.$$

Definition 5.1. Let $\mathbb{C}$ be a 2-category. The 2-category $\mathrm{Fr}^{2,1}\mathbb{C}$ has the same objects and arrows as $\mathbb{C}^2$, but a 2-cell between (u, φ, v) and $(w, \psi, x) : f \longrightarrow g$ is an equivalence class of 2-cells $\alpha : u \Rightarrow w$ for the equivalence relation

$$\alpha \sim \alpha' \text{ iff } g * \alpha = g * \alpha'.$$

Let $[\alpha]$ stand for the class of α. The compositions of 2-cells are easily defined: $[\alpha'] \circ [\alpha] = [\alpha' \circ \alpha]$ and $[\gamma] * [\alpha] = [\gamma * \alpha]$.

The 2-category $\mathrm{Fr}^{2,1}\mathbb{C}$ is equipped with a factorization system $(\mathcal{E}^{2,1}_{\mathbb{C}}, \mathcal{M}^{2,1}_{\mathbb{C}})$, which factorizes the arrows as in diagram (2).

Proposition 5.2. *The factorization system* $(\mathcal{E}^{2,1}_{\mathbb{C}}, \mathcal{M}^{2,1}_{\mathbb{C}})$ *in the 2-category* $\mathrm{Fr}^{2,1}\mathbb{C}$ *is (2,1)-proper.*

The 2-functor $P^{2,1}_{\mathbb{C}} : \mathbb{C}^2 \longrightarrow \mathrm{Fr}^{2,1}\mathbb{C}$ maps (α, β) to $[\alpha]$. We define $E^{2,1}_{\mathbb{C}} = P^{2,1}_{\mathbb{C}} \circ E_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}^2 \longrightarrow \mathrm{Fr}^{2,1}\mathbb{C}$.

Proposition 5.3. *For any 2-category* $\mathbb{C}$ *and for any 2-category with a (2,1)-proper factorization system* $(\mathbb{D}, (\mathcal{E}, \mathcal{M}))$*, the 2-functor*

$$- \circ E^{2,1}_{\mathbb{C}} : \mathrm{PS}_{fs}(\mathrm{Fr}^{2,1}\mathbb{C}, \mathbb{D}) \longrightarrow \mathrm{PS}(\mathbb{C}, \mathbb{D})$$

is a biequivalence.

Proof. As for Proposition 4.3, we have to prove that

$$- \circ P^{2,1}_{\mathbb{C}} : \mathrm{PS}_{fs}(\mathrm{Fr}^{2,1}\mathbb{C}, \mathbb{D}) \longrightarrow \mathrm{PS}_{fs}(\mathbb{C}^2, \mathbb{D})$$

is a biequivalence. The interesting part is, given a pseudo-functor $G : \mathbb{C}^2 \longrightarrow \mathbb{D}$ which preserves the factorization system, to construct a pseudo-functor $F : \mathrm{Fr}^{2,1}\mathbb{C} \longrightarrow \mathbb{D}$ which preserves the factorization system and such that $FP^{2,1}_{\mathbb{C}} \cong G$. We take $F = G$ on objects and arrows of $\mathrm{Fr}^{2,1}\mathbb{C}$. Consider now a 2-cell $[\alpha] : (u, \varphi, v) \Rightarrow (w, \psi, x) : f \longrightarrow g$ in $\mathrm{Fr}^{2,1}\mathbb{C}$. Define $\nu = \psi^{-1}(g * \alpha)\varphi : vf \Rightarrow xf$ (it is well-defined because we only use $g * \alpha$.) We get now a 2-cell ξ in the following way

$$\begin{array}{ccc} G(u,\varphi,v) \circ G(1_C, 1_f, f) & \overset{\cong}{\Longrightarrow} & G(u,\varphi,vf) \\ \Big\Downarrow \xi & & \Big\Downarrow G(\alpha,\nu) \\ G(w,\psi,x) \circ G(1_C, 1_f, f) & \overset{\cong}{\Longleftarrow} & G(w,\psi,xf) \end{array}$$

Since $(1_C, 1_f, f) \in \mathcal{E}_{\mathbb{C}}$ and G preserves the factorization system, $G(1_C, 1_f, f) \in \mathcal{E}$. Since $(\mathcal{E}, \mathcal{M})$ is (2,1)-proper, $G(1_C, 1_f, f)$ is fully cofaithful. This implies that there is a unique 2-cell $F([\alpha]) : G(u, \varphi, v) \Rightarrow G(w, \psi, x)$ such that

$$F([\alpha]) * G(1_C, 1_f, f) = \xi. \tag{6}$$

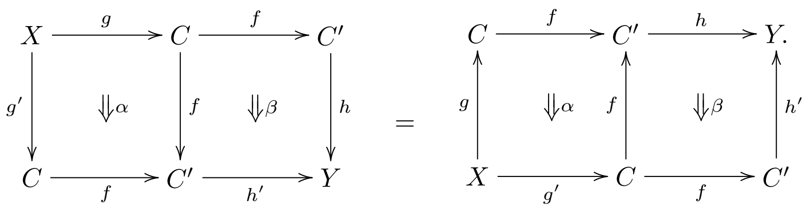
The argument to prove that F is well-defined is similar to that in the proof of Proposition 4.3.
Finally, if $(\alpha, \beta) : (u, \varphi, v) \Rightarrow (w, \psi, x) : f \longrightarrow g$ is a 2-cell in $\mathbb{C}^2$, then $F([\alpha]) = G(\alpha, \beta)$. For this, it is enough to check equation (6) for $G(\alpha, \beta)$. This follows from the fact that $\nu = \beta * f$. □

We can do exactly the same with (1,2)-proper factorization systems, and we get the free 2-category $E^{1,2}_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathrm{Fr}^{1,2}\mathbb{C}$. The difference is that, if $\mathbb{C}$ is a 2-category, the 2-cells of the 2-category $\mathrm{Fr}^{1,2}\mathbb{C}$ from (u, φ, v) to $(w, \psi, x) : f \longrightarrow g$ are the equivalence classes of 2-cells $\beta : v \Rightarrow x$ for the equivalence relation $\beta \sim \beta'$ iff $\beta * f = \beta' * f$.

6 (2,2)-proper factorization systems

The construction of $\mathrm{Fr}^{2,2}\mathbb{C}$, the free 2-category with a (2,2)-proper factorization system on a given 2-category $\mathbb{C}$, can be done if and only if the 2-category $\mathbb{C}$ is pre-full, in the sense of the following definition.

Definition 6.1. Let $\mathbb{C}$ be a 2-category, and let $f : C \longrightarrow C'$ be an arrow in $\mathbb{C}$. We say that f is *pre-full* if for each $g, g' : X \longrightarrow C$, for each $h, h' : C' \longrightarrow Y$, for each $\alpha : fg \Rightarrow fg'$ and for each $\beta : hf \Rightarrow h'f$, one has



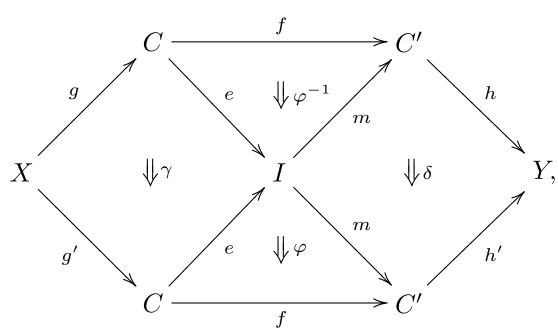
(7)

We say that $\mathbb{C}$ is *pre-full* if each arrow in $\mathbb{C}$ is pre-full.

The fact that any 2-category with a (2,2)-proper factorization system is pre-full follows immediately from the next lemma.

Lemma 6.2. Let $f : C \longrightarrow C'$ be an arrow in a 2-category $\mathbb{C}$ and consider an invertible 2-cell $\varphi : me \Rightarrow f$. If e and m are such that $- \circ e$ and $m \circ -$ are full functors, then f is pre-full.

Proof. Let us consider the situation of Definition 6.1. Let $\beta' = (h' {*} \varphi^{-1})\beta(h {*} \varphi) : hme \Rightarrow h'me$. Since $- \circ e$ is full, there exists $\delta : hm \Rightarrow h'm$ such that $\delta * e = \beta'$. In the same way, if $\alpha' = (\varphi^{-1} {*} g')\alpha(\varphi {*} g) : meg \Rightarrow meg'$, there exists $\gamma : eg \Rightarrow eg'$ such that $m * \gamma = \alpha'$, since $m \circ -$ is full. Then the 2 members of (7) are equal to the 2-cell



□

Let us explain now the reason why we can define $\mathrm{Fr}^{2,2}\mathbb{C}$ if and only if $\mathbb{C}$ is pre-full. We will define a 2-functor $E^{2,2}_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathrm{Fr}^{2,2}\mathbb{C}$ which is locally faithful. It is easy to see that this fact, together with the pre-fullness of $\mathrm{Fr}^{2,2}\mathbb{C}$ (which comes from its (2,2)-proper factorization system), implies that $\mathbb{C}$ is pre-full.

We arrive to the definition of $\mathrm{Fr}^{2,2}\mathbb{C}$.

Definition 6.3. Let $\mathbb{C}$ be a pre-full 2-category. The 2-category $\mathrm{Fr}^{2,2}\mathbb{C}$ has the same objects and arrows as $\mathbb{C}^2$, but a 2-cell from (u, φ, v) to $(w, \psi, x) : f \longrightarrow g$ is a 2-cell $\mu : gu \Rightarrow gw$. This is equivalent to give a 2-cell $\breve{\mu} : vf \Rightarrow xf$ related to μ by the equation $\breve{\mu} = \psi^{-1}\mu\varphi$. The vertical composition of $\mu : (u, \varphi, v) \Rightarrow (u', \varphi', v')$ (i.e. $\mu : gu \Rightarrow gu'$) and $\mu' : (u', \varphi', v') \Rightarrow (u'', \varphi'', v'')$ (i.e. $\mu' : gu' \Rightarrow gu''$) is simply $\mu' \circ \mu : gu \Rightarrow gu''$.

The horizontal composition is more problematic. Let $\mu : (u, \varphi, v) \Rightarrow (u', \varphi', v') : f \longrightarrow g$ and $\nu : (w, \psi, x) \Rightarrow (w', \psi', x') : g \longrightarrow h$. We define $\nu * \mu = (\psi' * u') \circ \tau_{\mu,\nu} \circ (\psi^{-1} * u) : hwu \Rightarrow hw'u'$, where $\tau_{\mu,\nu}$ is given by the following pasting

$$\begin{array}{ccccc} C & \xrightarrow{u} & D & \xrightarrow{g} & D' \\ {\scriptstyle u'}\downarrow & \Downarrow\mu & \downarrow{\scriptstyle g} & \Downarrow\breve{\nu} & \downarrow{\scriptstyle x} \\ D & \xrightarrow[g]{} & D' & \xrightarrow[x']{} & E'. \end{array}$$

One can check now that $\mathrm{Fr}^{2,2}\mathbb{C}$ is a 2-category: the pre-fullness of $\mathbb{C}$ is needed to prove the interchange law.

The 2-category $\mathrm{Fr}^{2,2}\mathbb{C}$ is equipped with a factorization system $(\mathcal{E}_{\mathbb{C}}^{2,2}, \mathcal{M}_{\mathbb{C}}^{2,2})$, which factorizes the arrows as in diagram (2).

Proposition 6.4. *The factorization system $(\mathcal{E}_{\mathbb{C}}^{2,2}, \mathcal{M}_{\mathbb{C}}^{2,2})$ in the 2-category $\mathrm{Fr}^{2,2}\mathbb{C}$ is (2,2)-proper.*

As in the previous sections, there is a 2-functor $P_{\mathbb{C}}^{2,2} : \mathbb{C}^2 \longrightarrow \mathrm{Fr}^{2,2}\mathbb{C}$ which is the identity on objects and 1-cells and maps $(\alpha, \beta) : (u, \varphi, v) \Rightarrow (u', \varphi', v') : f \longrightarrow g$ to $g * \alpha$. We define the 2-functor

$$E_{\mathbb{C}}^{2,2} : \mathbb{C} \longrightarrow \mathrm{Fr}^{2,2}\mathbb{C}$$

as the composite $P_{\mathbb{C}}^{2,2} \circ E_{\mathbb{C}}$. The next statement, which gives the universal property of $E_{\mathbb{C}}^{2,2}$, makes sense because a 2-category with a (2,2)-proper factorization system is pre-full.

Proposition 6.5. *For each pre-full 2-category $\mathbb{C}$ and for each 2-category with (2,2)-proper factorization system $(\mathbb{D}, (\mathcal{E}, \mathcal{M}))$, the 2-functor*

$$- \circ E_{\mathbb{C}}^{2,2} : \mathrm{PS}_{\mathrm{f}s}(\mathbb{C}^2, \mathbb{D}) \longrightarrow \mathrm{PS}(\mathbb{C}, \mathbb{D})$$

is a biequivalence.

Proof. Similar to that of Proposition 5.3. □

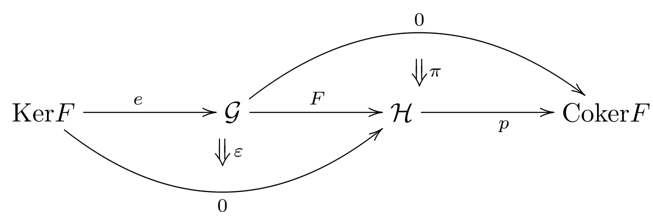
Remark: If the 2-category $\mathbb{C}$ is locally discrete, then it is pre-full and $\mathrm{Fr}^{2,2}\mathbb{C} = \mathrm{Fr}\mathbb{C}$ is the free category with proper factorization system studied in [15].

7 Examples and an open problem

7.1 Symmetric categorical groups

In [17], two examples of factorization systems are described in the 2-category SCG of symmetric categorical groups, monoidal functors preserving the symmetry and monoidal natural transformations. Let us set some notation. If

$F\colon \mathcal{G} \longrightarrow \mathcal{H}$ is a morphism in SCG, we write



for its kernel and its cokernel; we refer to [17] for their universal properties as bi-limits. If $\mathcal{G}$ is a symmetric cat-group, we write $\pi_0(\mathcal{G})$ for the abelian group of its connected components, and $\pi_1(\mathcal{G})$ for the abelian group $\mathcal{G}(I, I)$, where I is the unit object. If G is an abelian group, we write $D(G)$ for the discrete symmetric cat-group on G, and $G!$ for the symmetric cat-group with a unique object I, and such that $G!(I, I) = G$. These constructions have obvious extensions to morphisms.

In [17], it is proved that, by taking the kernel of the cokernel of an arrow in SCG, we get a factorization system $(\mathcal{E}_1, \mathcal{M}_1)$, where $\mathcal{E}_1$ is the class of full and essentially surjective functors, whereas $\mathcal{M}_1$ is the class of faithful functors. The second factorization system $(\mathcal{E}_2, \mathcal{M}_2)$ on SCG is obtained by taking the cokernel of the kernel of an arrow. In this case $\mathcal{E}_2$ is the class of essentially surjective functors and $\mathcal{M}_2$ is the class of fully faithful functors.

Proposition 7.1. *Let $F : \mathcal{G} \longrightarrow \mathcal{H}$ be an arrow in* SCG.

1. *F is faithful as an arrow in* SCG *if and only if F is faithful as a functor.*
2. *F is fully faithful as an arrow in* SCG *if and only if F is fully faithful as a functor.*
3. *F is cofaithful if and only if F is essentially surjective.*
4. *F is fully cofaithful if and only if F is full and essentially surjective.*

Proof. Only the necessary condition of 3. was not established in [17]. To prove this condition, let us recall that a functor F in SCG is essentially surjective if and only if $\pi_0 F$ is surjective.
Consider a cofaithful arrow $F : \mathcal{G} \longrightarrow \mathcal{H}$ in SCG. We have to prove that $\pi_0(F)$ is an epimorphism in the category Ab of abelian groups, i.e. for any $G \in$ Ab the mapping

$$- \circ \pi_0(F) : \mathrm{Ab}(\pi_0(\mathcal{H}), G) \longrightarrow \mathrm{Ab}(\pi_0(\mathcal{G}), G)$$

is surjective. Let us consider the one-object symmetric cat-group $G!$. There is a bijection

$$\varphi_{\mathcal{H}} : \mathrm{SCG}(\mathcal{H}, G!)(0, 0) \longrightarrow \mathrm{Ab}(\pi_0(\mathcal{H}), G)$$

which maps a monoidal natural transformation $\alpha : 0 \Rightarrow 0$ onto the group homomorphism $\varphi_{\mathcal{H}}(\alpha) : \pi_0(\mathcal{H}) \longrightarrow G : [X] \mapsto \alpha_X$. This map is well-defined because α is natural, and it is a group homomorphism because α is monoidal. The inverse of $\varphi_{\mathcal{H}}$ maps a morphism $f : \pi_0(\mathcal{H}) \longrightarrow G$ onto the natural transformation

$\varphi_{\mathcal{H}}^{-1}(f)$ such that $(\varphi_{\mathcal{H}}^{-1}(f))_X = f([X])$. In the same way, there is a bijection $\varphi_{\mathcal{G}} : \mathrm{SCG}(\mathcal{G}, G!)(0,0) \longrightarrow \mathrm{Ab}(\pi_0(\mathcal{G}), G)$. The announced result is immediate from the commutativity of the following diagram.

$$\begin{array}{ccc}
\mathrm{SCG}(\mathcal{H}, G!)(0,0) & \xrightarrow{-\circ F} & \mathrm{SCG}(\mathcal{G}, G!)(0,0) \\
\Big\downarrow{\scriptstyle \varphi_{\mathcal{H}}} & & \Big\downarrow{\scriptstyle \varphi_{\mathcal{G}}} \\
\mathrm{Ab}(\pi_0(\mathcal{H}), G) & \xrightarrow{-\circ \pi_0(F)} & \mathrm{Ab}(\pi_0(\mathcal{G}), G)
\end{array}$$

Indeed, the cofaithfulness of F implies that the top arrow is injective. Since the vertical arrows are bijective, this implies that the bottom arrow is injective. □

As a consequence, we have:

1. $(\mathcal{E}_1, \mathcal{M}_1)$ is a (2,1)-proper factorization system;
2. $(\mathcal{E}_2, \mathcal{M}_2)$ is a (1,2)-proper factorization system;
3. Let SCG^f be the sub-2-category of SCG whose arrows are the full functors; it is pre-full. Moreover, in SCG^f the systems $(\mathcal{E}_1, \mathcal{M}_1)$ and $(\mathcal{E}_2, \mathcal{M}_2)$ coincide and are (2,2)-proper.

From [17], we know that a morphism $F\colon \mathcal{G} \longrightarrow \mathcal{H}$ in SCG is essentially surjective iff it is the cokernel of its kernel $e\colon \mathrm{Ker}F \longrightarrow \mathcal{G}$. Moreover, there is a canonical morphism $c\colon \pi_1(\mathrm{Ker}F)! \longrightarrow \mathrm{Ker}F$, and F is full and essentially surjective iff it is the cokernel of the composite $e \circ c$. Therefore, we obtain the first factorization system taking the cokernel of $e \circ c$. Dually, F is faithful iff it is the kernel of its cokernel $p\colon \mathcal{H} \longrightarrow \mathrm{Coker}F$. There is a canonical arrow $d\colon \mathrm{Coker}F \longrightarrow D(\pi_0(\mathrm{Coker}F))$, and F is full and faithful iff it is the kernel of the composite $d \circ p$. Therefore, the second system can be obtained by taking the kernel of $d \circ p$.

We want now to describe the systems $(\mathcal{E}_1, \mathcal{M}_1)$ and $(\mathcal{E}_2, \mathcal{M}_2)$ using a different kind of bi-limits. We define the bi-limits we need in an arbitrary pointed 2-category.

Definition 7.2. Let $\mathbb{C}$ be a 2-category with a zero object 0 (that is, for any object $C \in \mathbb{C}$, the categories $\mathbb{C}(C, 0)$ and $\mathbb{C}(0, C)$ are equivalent to the one-arrow category).

1. Consider an arrow $f : C \longrightarrow C'$ in $\mathbb{C}$. The *pip* of f is given by an object $\mathrm{Pip}f$ and a 2-cell σ as in the following diagram,

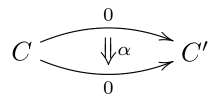
   $$\mathrm{Pip}f \underset{0}{\overset{0}{\rightrightarrows}} C \xrightarrow{f} C' \qquad (\Downarrow \sigma)$$

   such that $f * \sigma = f0$, and such that for any other

   $$X \underset{0}{\overset{0}{\rightrightarrows}} C \xrightarrow{f} C' \qquad (\Downarrow \chi)$$

   with $f * \chi = f0$, there is an arrow $t\colon X \longrightarrow \mathrm{Pip}f$, unique up to a unique invertible 2-cell, such that $\sigma * t = \chi$.

2. Consider a 2-cell



in $\mathbb{C}$. The *root* of α is an object $\mathrm{Root}\alpha$ and an arrow $r : \mathrm{Root}\alpha \longrightarrow C$ such that $\alpha * r = 0r$, and such that for any other $x : X \longrightarrow C$ with $\alpha * x = 0x$, there is $x' : X \longrightarrow \mathrm{Root}\alpha$ and an invertible 2-cell $\varphi : rx' \Rightarrow x$, the pair (x', φ) being unique up to a unique invertible 2-cell. (The root is a special case of *identifier*.)

3. The *copip* of f and the *coroot* of α are defined by the dual universal property.

We need an explicit description for the pip and the copip of a morphism in SCG. Let $F : \mathcal{G} \longrightarrow \mathcal{H}$ be an arrow in SCG.

1. The pip of F is given by $\mathrm{Pip}F = D(\mathrm{Ker}\pi_1(F))$ together with the monoidal natural transformation $\sigma : 0 \Rightarrow 0 : \mathrm{Pip}F \longrightarrow \mathcal{G}$ whose component at $\lambda \in \mathrm{Pip}F$ is λ.

2. The copip of F is given by $\mathrm{Copip}F = (\mathrm{Coker}\pi_0(F))!$ and by $\varrho : 0 \Rightarrow 0 : \mathcal{H} \longrightarrow \mathrm{Copip}F$, whose component at $X \in \mathcal{H}$ is $\varrho_X = [X]$, the equivalence class of X in $\mathrm{Coker}\pi_0(F)$, that is the isomorphism class of X in $\mathrm{Coker}F$.

Proposition 7.3. *Let $F : \mathcal{G} \longrightarrow \mathcal{H}$ be a morphism in* SCG.

1. *If F is fully cofaithful, then F is the coroot of its pip.*

2. *If F is fully faithful, then F is the root of its copip.*

Lemma 7.4. Let $\mathbb{C}$ be a pointed 2-category with pips and copips. Let $f : C \longrightarrow C'$ be an arrow in $\mathbb{C}$.

1. If $h : C' \longrightarrow Y$ is a faithful arrow, then $\mathrm{Pip}f = \mathrm{Pip}hf$.

2. If $g : X \longrightarrow C$ is a cofaithful arrow, then $\mathrm{Copip}f = \mathrm{Copip}fg$.

Proposition 7.5. *1. By taking the coroot of the pip of an arrow, we get the factorization system $(\mathcal{E}_1, \mathcal{M}_1)$.*

2. By taking the root of the copip of an arrow, we get the factorization system $(\mathcal{E}_2, \mathcal{M}_2)$.

Proof. Let $F : \mathcal{G} \longrightarrow \mathcal{H}$ be a morphism of symmetric cat-groups. Let $M_F \circ E_F$ be the $(\mathcal{E}_1, \mathcal{M}_1)$-factorization of F. Since E_F is fully cofaithful, E_F is the coroot of its pip, by Proposition 7.3. By Lemma 7.4, it is also the coroot of the pip of $F \cong M_F \circ E_F$, since M_F is faithful. So taking the coroot of the pip of F gives exactly its $(\mathcal{E}_1, \mathcal{M}_1)$-factorization. The proof of part 2 is similar. □

7.2 Categories

We discuss now some example in Cat, the 2-category of categories. Let us start by with a point of terminology.

Definition 7.6. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor.

1. F is *nearly surjective* (see [21]) if any $D \in \mathcal{D}$ is a retract of FC for some $C \in \mathcal{C}$.
2. F is *retract-stable* if for any $D \in \mathcal{D}$ which is a retract of FC for some $C \in \mathcal{C}$, there exists $C' \in \mathcal{C}$ such that $FC' \cong D$.

Clearly, a functor is essentially surjective on objects if and only if it is nearly surjective and retract-stable.

Example 7.7. The inclusion functor of a reflective subcategory is fully faithful and retract-stable.

Proposition 7.8. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor.*

1. *F is faithful in the sense of Definition 3.1 if and only if F is faithful in the usual sense.*
2. *F is fully faithful in the sense of Definition 3.1 if and only if F is fully faithful in the usual sense.*
3. *F is fully faithful and each $F \circ -$ is retract-stable if and only if F fully faithful and retract-stable*
4. *F is cofaithful if and only if F is nearly surjective.*
5. *If F is full and nearly surjective, F is fully cofaithful.*
6. *If F is full and essentially surjective, then F is fully cofaithul and each $- \circ F$ is retract-stable.*
7. *If F is full, then F is pre-full in the sense of Definition 6.1.*

Proof. Point 1, 2 and 3 are obvious. Point 4 is proved in [1]. Point 5 is proved in [17] in the 2-category SCG for full and essentially surjective functors; the proof for full and nearly surjective functors in Cat is an easy translation.
Let us prove point 6. If F is full and essentially surjective, by point 5., it is fully cofaithful. It remains to prove that each $- \circ F$ is retract-stable. For this, consider $G : \mathcal{D} \longrightarrow \mathcal{Y}$, $H : \mathcal{C} \longrightarrow \mathcal{Y}$, $\rho : GF \Rightarrow H$ and $\mu : H \Rightarrow GF$ such that $\rho \circ \mu = 1_H$. We define a functor $G' : \mathcal{D} \longrightarrow \mathcal{Y}$ in the following way. Given an object $D \in \mathcal{D}$, since F is essentially surjective there is $C_D \in \mathcal{C}$ and an invertible $\sigma_D : FC_D \longrightarrow D$. We put $G'D = HC_D$. If $f : D \longrightarrow D'$, consider the morphism

$$FC_D \xrightarrow{\sigma_D} D \xrightarrow{f} D' \xrightarrow{\sigma_{D'}^{-1}} FC_{D'}. \tag{8}$$

Since F is full, there exists $g_f : C_D \longrightarrow C_{D'}$ such that Fg_f is equal to the morphism (8). We put $G'f = Hg_f$.
The component at $C \in \mathcal{C}$ of the isomorphism $\omega : G'F \Rightarrow H$ is

$$G'FC = HC_{FC} \xrightarrow{\mu_{C_{FC}}} GFC_{FC} \xrightarrow{G\sigma_{FC}} GFC \xrightarrow{\rho_C} HC.$$

Its inverse is $\omega_C^{-1} =$

$$HC \xrightarrow{\mu_C} GFC \xrightarrow{G\sigma_{FC}^{-1}} GFC_{FC} \xrightarrow{\rho_{C_{FC}}} HC_{FC} = G'FC.$$

Finally, let us prove point 7. Consider two categories $\mathcal{X}, \mathcal{Y}$, four functors $G, G' : \mathcal{X} \longrightarrow \mathcal{C}$, $H, H' : \mathcal{D} \longrightarrow \mathcal{Y}$, and two natural transformations $\alpha : FG \Rightarrow FG'$ and $\beta : HF \Rightarrow H'F$. We have to prove that, for each $X \in \mathcal{X}$,

$$H'\alpha_X \circ \beta_{GX} = \beta_{G'X} \circ H\alpha_X. \tag{9}$$

Since F is full, there exists $f : GX \longrightarrow G'X$ such that $Ff = \alpha_X$. Equation (9) becomes now $H'Ff \circ \beta_{GX} = \beta_{G'X} \circ HFf$, which holds by naturality of β. $\square$

Let us also recall that fully cofaithful functors are characterized in two different ways in [1].

Example 7.9.

1. The first factorization system $\mathcal{S}_1$ is given by

$$\begin{array}{lcl} \mathcal{E}_1 & = & \{ \text{ full and essentially surjective functors} \} \\ \mathcal{M}_1 & = & \{ \text{ faithful functors } \} \end{array}$$

   A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ factors through $\mathrm{Im}_1 F$, which has the same objects as $\mathcal{C}$ and, if $C, C' \in \mathcal{C}$,

$$\mathrm{Im}_1 F(C, C') = F_{C,C'}(\mathcal{C}(C, C')).$$

   The composition is that of $\mathcal{D}$. By Proposition 7.8, this factorization system is (2,1)-proper.

2. The second factorization system $\mathcal{S}_2$ is given by

$$\begin{array}{lcl} \mathcal{E}_2 & = & \{ \text{ essentially surjective functors } \} \\ \mathcal{M}_2 & = & \text{ fully faithful functors } \} \end{array}$$

   A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ factors through $\mathrm{Im}_2 F$, which has the same objects as $\mathcal{C}$ and, if $C, C' \in \mathcal{C}$,

$$\mathrm{Im}_2 F(C, C') = \mathcal{D}(FC, FC').$$

   The composition is that of $\mathcal{D}$. By Proposition 7.8, this factorization system is (1,2)-proper.

3. The third factorization system $\mathcal{S}_3$ is given by

$$\begin{array}{lcl} \mathcal{E}_3 & = & \{ \text{ nearly surjective functors } \} \\ \mathcal{M}_3 & = & \{ \text{ retract-stable fully faithful functors } \} \end{array}$$

   A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ factors through $\mathrm{Im}_3 F$, which is a full subcategory of $\mathcal{D}$. An object is in $\mathrm{Im}_3 F$ if it is a retract of FC for some $C \in \mathcal{C}$. By Proposition 7.8, this factorization system is (1,2)-proper.

4. Here is a simple example of factorization system which is not (1,1)-proper. We write $\emptyset$ for the empty category.

$$\begin{array}{lcl} \mathcal{E}_4 & = & \{ \text{ the identity on } \emptyset \text{ and functors with non-empty domain } \} \\ \mathcal{M}_4 & = & \{ \text{ equivalences and functors with empty domain } \} \end{array}$$

The image of a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is $\mathcal{C}$ if $\mathcal{D} = \emptyset$, and $\mathcal{D}$ if $\mathcal{C} \neq \emptyset$.

5. As for SCG, let Cat^f be the sub-2-category of Cat of full functors. It is pre-full and $\mathcal{S}_1$ restricted to Cat^f is (2,2)-proper.

7.3 An open problem

Let us note that the first factorization system of Example 7.9 is not only (2,1)-proper but also "(3,1)-proper", in the sense that for any $E \in \mathcal{E}_1$, every composition functor $- \circ E$ is fully faithful and retract-stable. In the same way, the third factorization system is "(1,3)-proper", i.e. for any $M \in \mathcal{M}_3$, every composition functor $M \circ -$ is fully faithful and retract-stable. This suggests a more general definition of proper factorization system in a 2-category.

Definition 7.10. Let $\mathcal{S}_e = (\mathcal{E}_e, \mathcal{M}_e)$ and $\mathcal{S}_m = (\mathcal{E}_m, \mathcal{M}_m)$ be two factorization systems on the 2-category Cat. A factorization system $(\mathcal{E}, \mathcal{M})$ on a 2-category $\mathbb{C}$ is *$(\mathcal{S}_e, \mathcal{S}_m)$-proper* if

1. for any $e \in \mathcal{E}$, each composition functor $- \circ e$ belongs to $\mathcal{M}_e$;

2. for any $m \in \mathcal{M}$, each composition functor $m \circ -$ belongs to $\mathcal{M}_m$.

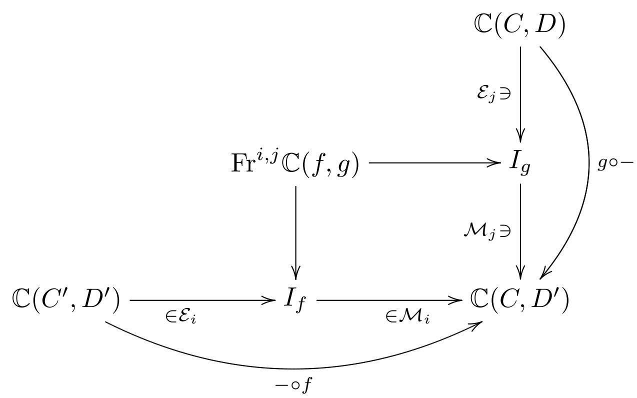
Following the notations of Subsection 7.2, we can reformulate Definition 3.2 in the following way:

> A factorization system on a 2-category $\mathbb{C}$ is (i, j)-proper exactly when it is $(\mathcal{S}_i, \mathcal{S}_j)$-proper, for $i, j \in \{1, 2\}$ (as well as for $(i, j) = (3, 1)$ and $(i, j) = (1, 3)$).

(Note that, if we put $\mathcal{S}_0 = (\text{equivalences}, \text{all arrows})$, every factorization system is $(\mathcal{S}_0, \mathcal{S}_0)$-proper.)

Observe that the free 2-category with (i,j)-proper factorization system $\mathrm{Fr}^{i,j}\mathbb{C}$ on a 2-category $\mathbb{C}$, for $i, j \in \{1, 2\}$ (Sections 4, 5 and 6), can be described in the following way.

Let $f : C \longrightarrow C'$ and $g : D \longrightarrow D'$ be in $\mathbb{C}$; consider the $\mathcal{S}_i$-factorization of the functor $- \circ f : \mathbb{C}(C', D') \longrightarrow \mathbb{C}(C, D')$ and the $\mathcal{S}_j$-factorization of the functor $g \circ - : \mathbb{C}(C, D) \longrightarrow \mathbb{C}(C, D')$. Then the hom-category $\mathrm{Fr}^{i,j}\mathbb{C}(f, g)$ is given by the following bi-pullback in Cat:



(This is the case also for $(i,j) = (0,0)$, where $\mathrm{Fr}^{0,0}\mathbb{C}$ is simply the 2-category $\mathbb{C}^2$ of Section 2.)

The problem arising from this remark is if it is possible to generalize the previous construction to get the free 2-category with $(\mathcal{S}_e, \mathcal{S}_m)$-proper factorization system on $\mathbb{C}$. To define the composition functor on the hom-categories, further assumptions on $\mathbb{C}$ are needed (as the example $\mathrm{Fr}^{2,2}\mathbb{C}$ shows), but we are not able to state them explicitly.

8 A glance at the homotopy category

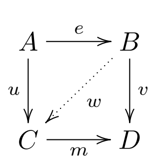
Recall that a weak factorization system in a category $\mathcal{C}$ consists of two classes of morphisms $(\mathcal{E}, \mathcal{M})$ satisfying the following conditions:

1) Given three arrows $A \xrightarrow{f} B \xrightarrow{i} X \xrightarrow{p} B$, if $i\circ f \in \mathcal{E}$ and $p\circ i = 1_B$, then $f \in \mathcal{E}$;

2) Given three arrows $A \xrightarrow{j} X \xrightarrow{q} A \xrightarrow{f} B$, if $f \circ q \in \mathcal{M}$ and $q \circ j = 1_A$, then $f \in \mathcal{M}$;

3) Each arrow has a $(\mathcal{E}, \mathcal{M})$-factorization;

4) Given a commutative square

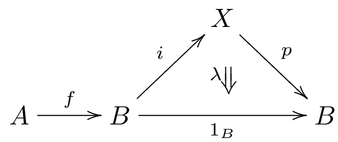


if $e \in \mathcal{E}$ and $m \in \mathcal{M}$, then there is a (not necessarily unique) arrow $w \colon B \to C$ such that $w \circ e = u$ and $m \circ w = v$.

The aim of this section is to show that a factorization system in a 2-category $\mathbb{C}$ induces a weak factorization system in the homotopy category $H(\mathbb{C})$ of $\mathbb{C}$ (the category $H(\mathbb{C})$ has the same objects as $\mathbb{C}$, and 2-isomorphism classes of 1-cells as arrows). The main fact is stated in the following proposition.

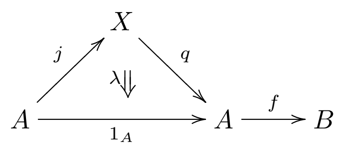
Proposition 8.1. *Let $\mathbb{C}$ be a 2-category with a factorization system $(\mathcal{E},\mathcal{M})$.*

1) Consider the following diagram



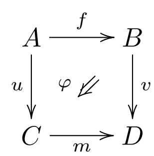
if λ is invertible and $i \circ f \in \mathcal{E}$, then $f \in \mathcal{E}$;

2) Consider the following diagram



if λ is invertible and $f \circ q \in \mathcal{M}$, then $f \in \mathcal{M}$.

Proof. We prove the first part, the second one is similar. We have to show that $f \in \mathcal{M}^{\uparrow}$. For this, we check the first condition in Proposition 1.4 and we leave the second one to the reader. Consider the following diagram in $\mathbb{C}$, with $m \in \mathcal{M}$,



We get an arrow $(u, \varphi \circ (v * \lambda * f), vp) \colon if \longrightarrow m$ with a universal fill-in (α, w, β) (because $if \in \mathcal{E}$ and $m \in \mathcal{M}$). This fill-in gives rise to a fill-in (α, wi, γ) for $(u, \varphi, v) \colon f \longrightarrow m$, where $\gamma = (v * \lambda) \circ (\beta * i)$, and we have to prove that (α, wi, γ) is universal. Let (α', w', β') be another fill-in for (u, φ, v). We get a second fill-in $(\alpha' \circ (w' * \lambda * f), w'p, \beta' * p)$ for $(u, \varphi \circ (v * \lambda * f), vp)$, so that there is a unique comparison $\psi \colon w \Rightarrow w'p$. This gives us a comparison $\mu = (w' * \lambda) \circ (\psi * i) \colon wi \Rightarrow w'$ between the two fill-in for (u, φ, v), and we have to prove that such a comparison is unique. Let $\overline{\mu} \colon wi \Rightarrow w'$ be another comparison between the two fill-in for (u, φ, v). Observe that $(\alpha \circ (wi * \lambda * f), wip, \gamma * p)$ is a third fill-in for $(u, \varphi \circ (v * \lambda * f), vp)$, so that there is a unique comparison $\nu \colon wip \Rightarrow w'p$ between $(\alpha \circ (wi * \lambda * f), wip, \gamma * p)$ and $(\alpha' \circ (w' * \lambda * f), w'p, \beta' * p)$ (because, by Lemma 1.5, each fill-in is universal). A diagram chasing shows that both $\nu = \mu * p$ and $\nu = \overline{\mu} * p$ work, so that $\mu * p = \overline{\mu} * p$. Finally, observe that, since $\lambda \colon pi \Rightarrow 1_B$ is an invertible 2-cell, p is a cofaithful (that is, the hom-functor

$$\mathbb{C}(p, C) \colon \mathbb{C}(B, C) \longrightarrow \mathbb{C}(X, C)$$

is faithful). Now $\mu * p = \overline{\mu} * p$ implies $\mu = \overline{\mu}$. □

In the next corollary, we write $[f]$ for the 2-isomorphism class of an arrow f.

Corollary 8.2. *Let $\mathbb{C}$ be a 2-category with a factorization system $(\mathcal{E},\mathcal{M})$ and let $H(\mathbb{C})$ be the homotopy category of $\mathbb{C}$. Then $(H(\mathcal{E}), H(\mathcal{M}))$ is a weak factorization system in $H(\mathbb{C})$, where $H(\mathcal{E}) = \{[e] \mid e \in \mathcal{E}\}$ and $H(\mathcal{M}) = \{[m] \mid m \in \mathcal{M}\}$.*

References

[1] J. Adamek, R. El Bashir, M. Sobral, J. Velebil: *On functors which are lax epimorphisms*, Theory Appl. Categories 8 (2001) 509-521 (http://www.tac.mta.ca/tac/).

[2] J. Adamek, H. Herrlich, J. Rosicky, W. Tholen: *Weak factorization systems and topological functors*, Appl. Categorical Structures (to appear).

[3] M.A. Bednarczyk, A.M. Borzyszkowski, W. Pawlowski: *Generalized congruences – epimorphisms in* Cat, Theory Appl. Categories 5 (1999) 266-280 (http://www.tac.mta.ca/tac/).

[4] T. Beke: *Sheafifiable homotopy model categories*, Math. Proc. Camb. Philos. Soc. 129 (2000) 447-475.

[5] A. Beligiannis: *On the Freyd categories of an additive category*, Homology Homotopy Appl. 2 (2000) 147-185 (http://www.rmi.acnet.ge/hha/).

[6] R. Betti, D. Schumacher, R. Street: *Factorizations in bicategories*, preprint.

[7] F. Borceux: *Handbook of categorical algebra*, Cambridge University Press (1994).

[8] F. Borceux, G. Janelidze: *Galois theories*, Cambridge University Press (2001).

[9] A. Carboni, S. Johnson, R. Street, D. Verity: *Modulated bicategories*, J. Pure Appl. Algebra 94 (1994) 229-282.

[10] J.M. Cohen: *Stable homotopy*, Springer Lecture Notes Math. 194 (1970).

[11] D. Dikranjan, W. Tholen: *Categorical structure of closure operators. With applications to topology, algebra and discrete mathematics*, Kluwer Academic Publishers (1995).

[12] M. Dupont: *Systèmes de factorisation dans les 2-catégories*, Mémoire de Licence, Université catholique de Louvain (2001).

[13] P. Freyd: *Stable homotopy*, Proc. Conf. Categor. Algebra, La Jolla 1965 (1966) 121-172.

[14] M. Grandis: *Weak subobjects and the epi-monic completion of a category*, J. Pure Appl. Algebra 154 (2000) 193-212.

[15] M. Grandis: *On the monad of proper factorization systems in categories*, J. Pure Appl. Algebra (to appear).

[16] M. Hovey: *Model categories*, American Mathematical Society (1999).

[17] S. Kasangian, E.M. Vitale: *Factorization systems for symmetric cat-groups*, Theory Appl. Categories 7 (2000) 47-70 (http://www.tac.mta.ca/tac/).

[18] P.T. Johnstone: *Factorization theorems for geometric morphisms, I*, Cahiers Topologie G.D. 22 (1981) 1-15.

[19] G.M. Kelly, R. Street: *Review of the elements of 2-categories*, Springer Lecture Notes Math. 420 (1974) 75-103.

[20] M. Korostenski, W. Tholen: *Factorization systems as Eilenberg-Moore algebras*, J. Pure Appl. Algebra 85 (1993) 57-72.

[21] S. Lack, E. M. Vitale: *When do completion processes give rise to extensive categories?* J. Pure Appl. Algebra 159 (2001) 203-230.

[22] S. Milius: *Factorization systems in 2-categories*, preprint (2001).

[23] A. Neeman: *Triangulated categories*, Princeton University Press (2001).

[24] J. Rosicky, W. Tholen: *Lax factorization algebras*, J. Pure Appl. Algebra (to appear).

[25] J. Rosicky, E.M. Vitale: *Exact completion and representations in abelian categories*, Homology Homotopy Appl. 3 (2001) 453-466 (http://www.rmi.acnet.ge/hha/).

[26] R. Street: *Two-dimensional sheaf theory*, J. Pure Appl. Algebra 23 (1982) 251-270.

Département de Mathématiques,
Université catholique de Louvain
Chemin du Cyclotron 2,
1348 Louvain-la-Neuve, Belgium.
e.mail : dupont@math.ucl.ac.be, vitale@math.ucl.ac.be