Categories without a set of objects, n-orders, constructivism

Mathieu Dupont

2008

(This was intended to be a warning at the beginning of my PhD thesis [9], entitled "Warning concerning the framework of this work" but I removed it at the last minute.)

The goal of this "warning" is to explain the framework in which this work has been conceived and carried out. This framework is reflected in notations, in terminology, as well as in some directions followed in this work.

1 Sets

Zermelo-Fraenkel theory is often presented as a theory in which it is possible to develop mathematics. But this theory ZF has two main shortcomings: on the one hand, mathematics are not directly formalised in ZF, they have to be *encoded* by hand more or less arbitrarily; on the other hand, through this encoding, this theory allows the expression of properties without mathematical meaning, like

- $1 \in 2$,
- $\cos \in \pi$, or
- $(\mathbb{N} \times \mathbb{N}) \cap \mathcal{P}(\mathbb{N}) = (1, 2)$.

This second problem is due to the fact that mathematics are *typed* (i.e. ordinary mathematical functions and relations are each defined for mathematical objects of a given type, and their application to objects of another type has no meaning), whereas ZF is an untyped theory.

That is why this work is not developed in ZF, but in some theory of types (which could look like Intuitionistic Type Theory by Martin-Löf [18] or like the Calculus of Constructions [8]). We can think of a *type* as a simple collection of objects, without any structure, not even an equality relation between objects.

That is what Errett Bishop calls a preset [7]. To describe a type, it is enough to describe what an object consists of.

We use the notation "$x: A$" to declare that a variable x is of type A (we use this notation also in quantifications, for example: $\forall x: A. P(x)$). We will reserve the symbol $\in$ for the membership relation between elements of a set A and subsets of A, or between objects of a category $\mathbb{A}$ and full subcategories of $\mathbb{A}$.

A set is defined as a type equipped with an equality relation.

1 Definition. A *set* is a type A equipped, for all objects $a, b: A$, with a truth value $a = b$, such that, for all $a, b, c: A$,

1. $\top \leq (a = a)$;
2. $(a = b) \wedge (b = c) \leq (a = c)$;
3. $(a = b) \leq (b = a)$.

This is the definition of set used in constructive mathematics by Errett Bishop[7]. It is also often used for the formalisation of mathematics in proof assistants (sometimes with the name "setoid"); for example, in [1], [19] and [11]. These sets are essentially the abstract sets in the sense of Lawvere [13, 14], which are the objects of the category of sets.

More generally, one can define an *order* as a type A equipped with such a structure, without condition 3 (we use then the notation $a \leq b$ rather than $a = b$). There is no question of antisymmetry, because there are no preexisting equality allowing to express it. On the other hand, we can define an equality on the order A *a posteriori*, by defining $a = b$ by the condition: $a \leq b$ and $b \leq a$.

A rule mapping each element x of type A to an element $f(x)$ of type B will be called an *operation* (in the sense of Bishop [7]). A function is an operation preserving the equality. That is the definition used by Bishop and the above-mentioned formalisations of mathematics.

2 Definition. Let A and B be sets. A *function* $f: A \to B$ is given, for each object $a: A$, by an objet $f(a): B$ such that, if $a = b$, then $f(a) = f(b)$.

The functions from A to B form a set $\mathsf{Set}(A, B)$, whose equality is defined pointwise: $f = g$ if, for each $a: A$, $f(a) = g(a)$. For each set A, there is an identity function $1_A: A \to A$ and, if we have functions $A \xrightarrow{f} B \xrightarrow{g} C$, there is a composite function $A \xrightarrow{gf} B$, defined in the usual way.

The quotient of a set A by an equivalence relation R can be described very simply: it is the set which has the same objects as A and whose equality relation is R. It's the same principle as for quotients of categories: when we quotient a category, we keep the same objects and we add isomorphisms (or, more generally, morphisms) to identify some objects; in the case of sets, to identify two elements, we add an equality between these elements. It is not necessary to use equivalence classes. In this way we avoid making arbitrary choices of a representing element of an equivalence class, and this is closer to ordinary mathematical practice with rational numbers or integers modulo n.

2 Constructivism and axiom of choice

This work will be developed constructively. In particular, the assertion of the existence of an object having a certain property will mean that we possess an effective construction of an object having this property. The assertion of $\forall x\colon A.\, \exists y\colon B.\, P(x,y)$ means that we do have a rule f mapping each object x of type A to an objet $f(x)$ of type B such that $P(x, f(x))$; we have thus an operation $f\colon A \to B$.

For example, we say that a function $f\colon A \to B$ is a *surjection* if, for every $b\colon B$, there exists $a\colon A$ such that $f(a) = b$. This means that we have an operation $g\colon B \to A$ such that, for each $b\colon B$, $f(g(b)) = b$. In general, there is no reason to assume that this operation g preserves equality, i.e. that g is a function. That is why we won't assume that the axiom of choice is true for sets. On the other hand, it is easy to prove the *axiom of unique choice*, in the following form.

3 PROPOSITION. *Every bijection (surjective and injective function) $A \to B$ is invertible.*

PROOF. Let $f\colon A \to B$ be a bijection. We have thus an operation $g\colon B \to A$ such that, for each $b\colon B$, $fg(b) = b$. Then g is a function (it preserves equality): if $b = b'$ in B, $fgb = b = b' = fgb'$ and so, by the injectivity of f, $gb = gb'$. It only remains to prove that this function is an inverse for f. We already know that $fg = 1_B$. Moreover, $gf = 1_A$ since, for each $a\colon A$, $fgfa = fa$ and, since f is injective, $gfa = a$. $\square$

3 Categories

We cannot define an equality on the type of sets by comparing the elements of two sets, because equality is defined only between elements of the same set. The type of all sets has no natural structure of set. On the other hand, this type is a category. The structure of category is a structure similar to that of set, or better to that of order, as Lawvere has noticed [12].

4 DEFINITION. A *category* is a type $\mathbb{A}$ equipped with:

1. for all objects $A, B\colon \mathbb{A}$, a set $\mathbb{A}(A,B)$ (also denoted by $A \to B$);

2. for each object $A\colon \mathbb{A}$, an object $1_A\colon A \to A$;

3. for all objects $A, B, C\colon \mathbb{A}$, a function $\mathrm{comp}\colon \mathbb{A}(A,B) \times \mathbb{A}(B,C) \to \mathbb{A}(A,C)$, mapping $A \xrightarrow{f} B \xrightarrow{g} C$ to $g \circ f\colon A \to C$.

This must satisfy the following conditions:

1. $f \circ 1_A = f = 1_B \circ f$;

2. $(h \circ g) \circ f = h \circ (g \circ f)$.

The essential difference with the usual definition of a category is that we do not assume that the type of the objects of a category have, besides the strictly categorical structure, a structure of set (i.e. an equality at the level of objects). It is indeed natural to consider these structures separately before combining them.

One of the general principles of usual category theory is that one should avoid speaking of equality between objects, the true identity criterion for objects in a category being isomorphism. It is thus natural to take into account this principle in the very definition of category by making simply impossible the expression of such an equality.

This point of view has been defended by Mihaly Makkai, in the first sections of the paper *Towards a categorical foundation of mathematics* [16], and supported by Jean-Pierre Marquis [17]. The possibility of considering categories without equality at the level of objects was also considered, in a different context, by Jean Bénabou [6]. In constructive mathematics, this is common: Roger Apéry [2] already did consider that equality between objects of a category has no meaning, and the formalisations of category theory in type theories usually do not include an equality at the level of objects [1, 19, 11].

An advantage of not requiring an equality between objects is that sets will form a category, without size limitations on sets, since the objects of the category are not required to form themselves a set. Moreover, Freyd's theorem ([10, exercice 3.D]) asserting that a "completely" complete category is necessarily an order won't apply, because it uses the set of arrows, which doesn't exist for the more general notion of category defined above. It is nevertheless necessary to introduce some kind of limitation on the possible constructions because, if both the covariant powerset functor $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$ and all colimits in Set exist, we could prove (by a categorical version of Knaster-Tarski theorem) that this functor $\mathcal{P}$ has a fixed point, which is impossible by Cantor theorem. Since in this work we do not get close to the danger zone, we won't settle this question by fixing a limitation.

Sets can be seen as categories. An advantage of the definitions used here is that if a category is equivalent to a set, it is a set; the notion of set is thus a categorical notion, in the sense "invariant under equivalence".

Functors $\mathbb{A} \to \mathbb{B}$ are the operations preserving the category structure (in the same way as functions are the operations preserving the set structure, i.e. equality). We will say that a functor $F\colon \mathbb{A} \to \mathbb{B}$ is *surjective* if for every $B\colon \mathbb{B}$ there exists an object $G(B)\colon \mathbb{A}$ and an isomorphism $\varphi_B\colon FGB \to B$. As in the case of surjective functions, an operation $G\colon \mathbb{B} \to \mathbb{A}$ is explicitly given, by constructivity. We can then prove an *axiom of unique choice* for categories.

5 PROPOSITION. *Every faithful, full and surjective functor is an equivalence.*

PROOF. Starting with the definition of surjective functor given above, it is enough to prove that the operation G is a functor: we have to define its action on arrows. Let $B, B'\colon \mathbb{B}$. We set $G_{B,B'} :=$

$$\mathbb{B}(B, B') \xrightarrow{\varphi_{B'}^{-1} \circ - \circ \varphi_B} \mathbb{B}(FGB, FGB') \xrightarrow{F^{-1}_{GB,GB'}} \mathbb{A}(GB, GB'), \tag{1}$$

where $F^{-1}_{GB,GB'}$ is given by Proposition 3 applied to $F_{GB,GB'}$ (which is injective by the faithfulness of F and surjective by its fullness). It is easy to check that this defines an inverse for F. □

4 Categories with a set of objects

The rôle played in standard category theory by the quantitative distinction between small and large categories is played here by the qualitative distinction between categories with or without a set of objects. The category of sets is an example of category without a set of objects (a "large" category).

Some results, using explicitly or implicitly an equality at the level of objects, will be restricted to the case of categories equipped with a set of objects. For example, the caracterisation of limits in terms of products and equalizers has a meaning only for limits of a functor whose domain has a set of objects (and thus of arrows), because it is constructed by products indexed by the set of objects and the set of arrows of this domain.

Most of the works about groupoids and groupoid enriched categories concern internal groupoids in Set, which form a category $\mathsf{Gpd}(\mathsf{Set})$. Since in this work we want to study the 2-category of groupoids, we have to work with general groupoids, i.e. not having necessarily a set of objects, so as to avoid the use of the axiom of choice. We could work instead with internal anafunctors and natural anatransformations [15, 4], but it is easier to work directly with general groupoids.

5 n-categories and coherence

We noticed that sets do not form a set, but a category. In the same way, categories do not form a category (this would require an equality between functors and so an equality between objects in categories), but a 2-category (see again [16] and [17]). In the same way, 2-categories do not form a 2-category, but a 3-category and so on.

Apparently, categories form a category: the associativity of the composition of functors is "strict": if we compute $((F \circ G) \circ H)(C)$ and $(F \circ (G \circ H))(C)$, we get in both cases $F(G(H(C)))$. But in the absence of an equality at the level of objects, this can not be expressed in the language of category theory. This equality is in fact a syntactical equality, in the metalanguage, between the expressions "$((F \circ G) \circ H)(C)$" and "$(F \circ (G \circ H))(C)$", after unfolding the definition of composite functor. This is what is often called in type theory "intensional equality" or "definitional equality" (the equality of the language (which exists here only between arrows from a fixed object A to a fixed object B) is called "extensional equality").

We adopt here the point of view that the results called "coherence results", expressing this kind of strictness, are metatheorems asserting the existence of an equivalent description of a category, or of a 2-category, in which certain isomorphism can be taken as identities. We will denote by $\equiv$ the (definitional) syntactical equality: $(F \circ G) \circ H \equiv F \circ (G \circ H)$.

The great advantage of coherence results is that they simplify calculations and reduce the size of diagrams. But we won't consider here that there are on the one hand weak 2-categories (or bicategories [5]) and, on the other hand, strict 2-categories; the definition we use will be the "weak" one, which is the only one available since there is no equality at the level of objects and arrows of a 2-category. The strict version is a simplified description made possible by a coherence theorem. Bicategories have been formalised, in the system `Agda`, by Olov Wilander [19].

This explains why the default definitions of 2-categorical notions will be here the weakest possible (using equality only at the level of 2-arrows). But we use, when they are useful, strictified descriptions of these notions. This has an implication with regard to terminology: we won't burden ourselves with the prefixes "pseudo-" or "bi-", because they are not useful in the absence of rival notions. When a 2-dimensional notion restricts to the corresponding 1-dimensional notion in a category seen as a locally discrete 2-category (in this case the 2-dimensional notion is a generalisation of the 1-dimensional one), we use the same name for the 2-dimensional and 1-dimensional notions. But we add the prefix "2-" when the 2-dimensional notion is an analogue of the 1-dimensional one, but is not a generalisation. For example, 2-relations are the 2-dimensional analogue of relations in categories, but in a category the 2-relations are not relations (but spans, not necessarily jointly monorphic), so the prefix “2-” is necessary; on the other hand, the definition of the kernel of an arrow in dimension 2 becomes the usual kernel in dimension 1, so we don't use the prefix “2-”. Would we work in the 3-category of internal 2-categories, the presence of equality at the level of objects and arrows would permit to distinguish different levels of strictness; in this case, we use the word without prefix for the weakest case, to be coherent with the above convention, and we add the adjective “strict” to name the case where the coherence 2-cells are replaced by equalities.

6 Dimension of higher-order categorical structures

Higher-dimensional category theory is usually presented in the following way: in dimension 1, we study 1-categories (Set-categories) [which are often categories of sets with structure], in dimension 2, we study 2-categories (Cat-categories) [which are often 2-categories of categories with structure], in dimension 3 we study 3-categories, and so on. We have thus a hierarchy

$$\mathsf{Set} \hookrightarrow \mathsf{Cat} \hookrightarrow 2\text{-}\mathsf{Cat} \hookrightarrow \ldots \tag{2}$$

But this point of view has two shortcomings.

1. Orders are categories but not sets; they would be thus classified as being of higher dimension than sets and of the same dimension as categories. But the reality is very different: in orders like in sets all arrows are equal (all diagrams commute), whereas in categories there can be as many arrows as you want between two objects, there is thus one more degree of freedom than in the case of orders and sets. This further degree of freedom is the origin of specifically 2-dimensional phenomena: when we define algebraic structures on categories, the arrows expressing associativity, commutativity, etc., have to satisfy coherence conditions. A second specifically 2-dimensional phenomenon is the splitting of ordinary notions: there are two kinds of monomorphisms and epimorphisms, two kinds of kernels, two factorisations of arrows, etc. We will get back to this later. Orders are not affected by these phenomena.

2. Another problem with this standard notion of dimension is that when going from sets to categories (and so from Set-categories to Cat-categories), there are two changes at the same time: a dimension change, but also a change from a groupoidal case to a non-groupoidal case. And some differences which are in fact due to the loss of symmetry are sometimes attributed to the higher dimension (for example the distinction between pullback and

comma object). It is thus preferable to clearly separate these two modifications. In this work, we increase the dimension by one, but we stay in the symmetric (groupoidal) case.

We say thus that a categorical structural (to be think of as an ω-category) is of dimension n if for all $k > n+1$, all k-arrows between two $(k-1)$-arrows are equal. This is equivalent to require that, for each pair of $(k-1)$-arrows α, β, there be exactly one k-arrow (up to equivalence) from α to β. In a n-dimensional ω-category, we have thus at most one $(n+1)$-arrow (up to isomorphism) between two n-arrows (because they are all equal).

For $n = 0$, we get thus orders, for $n = 1$, Ord-categories (or 2-orders), for $n = 2$, $(2\text{-}\mathsf{Ord})$-categories (or 3-orders), and so on. John Baez, Toby Bartels et James Dolan have noticed that we can also attribute negative values to n (see [3]). For $n = -1$, we can have at most one 0-arrow, i.e. at most one object. We can call (-1)-orders *truth values*; the empty (-1)-order is the truth value “false” and the one-object (-1)-order is the truth value “true”. We will denote (-1)-Ord by Ω. For $n = -2$, the definition has also a meaning, when we take it in the form “for $k > n+1$, for each pair of $(k-1)$-arrows α, β, there is exactly one k-arrow from α to β”: there must be, for $k > -1$, i.e. $k \geq 0$ exactly one k-arrow between each pair of $(k-1)$-arrows: for $k = 0$, this means that there must be exactly one object in the ω-category, with one arrow on this object, one 2-arrow, and so on. It is thus 1, and (-2)-$\mathsf{Ord} = 1$ (this seems to be an example of noncontradictory self-reflexivity, another being the fact that ω-Cat[1] is an ω-category). The fundamental hierarchy is thus rather:

$$1 \hookrightarrow \Omega \hookrightarrow \mathsf{Ord} \hookrightarrow 2\text{-}\mathsf{Ord} \hookrightarrow \ldots, \tag{3}$$

where $\Omega = 1$-Cat, $\mathsf{Ord} = 2$-Cat and so on. This will be the framework of reference for this text.

At each level, groupoids play a special rôle. At the first two levels, everything is a groupoid. At level 3, the groupoids are the sets, and at the following level, they are the usual groupoids. This work is concerned with a part of 2-dimensional algebra, which generalises usual algebra on sets. It will be about algebra on “2-sets”, i.e. on groupoids.

[1] which should rather be called ω-Ord

All this gives the following diagram, which can be found also in [3].

$$\begin{tikzcd}
\mathsf{2\text{-}Gpd} \arrow[r] & \mathsf{Gpd\text{-}Cat} \arrow[r] & \mathsf{2\text{-}Cat} \arrow[r] & \mathsf{3\text{-}Ord} \\
& \mathsf{Gpd} \arrow[ul] \arrow[u] \arrow[r] & \mathsf{Cat} \arrow[u] \arrow[r] & \mathsf{2\text{-}Ord} \arrow[u] \\
& & \mathsf{Set} \arrow[ul] \arrow[u] \arrow[r] & \mathsf{Ord} \arrow[u] \\
& & & \Omega \arrow[ul] \arrow[u] \\
& & & 1 \arrow[u]
\end{tikzcd} \tag{4}$$

The rôle of ordinary categories in this scheme is that sets with structures form Set-categories. The rôle played in dimension 1 by sets being played in dimension 2 by groupoids, the relevant level of generality for this work is that of Gpd-categories, and not that of Cat-categories (2-categories). That is why we talk systematically about Gpd-categories (another reason is that the Gpd-category of groupoids has better properties, with respect to factorisations, than the 2-category of categories).

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